Recursive solutions for Laplacian spectra and eigenvectors of a class of growing treelike networks

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The complete knowledge of Laplacian eigenvalues and eigenvectors of complex networks plays an outstanding role in understanding various dynamical processes running on them; however, determining analytically Laplacian eigenvalues and eigenvectors is a theoretical challenge. In this paper, we study the Laplacian spectra and their corresponding eigenvectors of a class of deterministically growing treelike networks. The two interesting quantities are determined through the recurrence relations derived from the structure of the networks. Beginning from the rigorous relations one can obtain the complete eigenvalues and eigenvectors for the networks of arbitrary size. The analytical method opens the way to analytically compute the eigenvalues and eigenvectors of some other deterministic networks, making it possible to accurately calculate their spectral characteristics.

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I. INTRODUCTION

As an interdisciplinary subject, complex networks have received tremendous recent interest from the scientific community [1–5] because of their flexibility and generality in the description of natural and manmade systems. A central issue in the study of complex networks is to understand how their dynamical behaviors are influenced by the underlying topological structure [3–5]. In various dynamical processes, the effect of network structure is encoded in the eigenvalues (spectra) and their corresponding eigenvectors of its Laplacian matrix. For instance, the synchronizability of a network is determined by the ratio of the maximum eigenvalue to the smallest nonzero one of its Laplacian matrix [6,7]. Again, for example, for continuous-time quantum walks [8] in a network, the quantum transition probabilities [9] between two nodes are closely related to the eigenvalues and orthonormalized eigenvectors of its Laplacian matrix, which also determine the resistance between a pair of nodes and the average resistance of all couples of nodes in a resistance network [10,11]. Thus, the complete (exact) knowledge of Laplacian spectra and eigenvectors is very important for understanding the network dynamics.

Recently, a lot of activities have been devoted to the study of the spectra of complex networks [12–15], providing useful insight into the topological properties of and dynamical processes on networks. However, most previous related studies have been confined to approximate or numerical methods, the latter of which is prohibitively difficult for large networks because of the limit of time and memory. Moreover, notwithstanding its significance, relevant research on eigenvectors of Laplacian matrix of complex networks is much less.

In the present paper, we investigate the Laplacian eigenvalues and eigenvectors of a class of deterministic treelike networks, which are constructed iteratively [16]. By applying the technique of graph theory and an algebraic iterative procedure, we derive recursive relations for the Laplacian eigenvalues and eigenvectors of the networks. The obtained recurrence relations allow one to determine explicitly the full Laplacian eigenvalues and eigenvectors of the considered networks of arbitrary iterations from those of its initial structure.

II. MODEL FOR THE GROWING TREES

Here we introduce a model for a class of deterministically growing trees (networks) defined in an iterative way [16], which has attracted an amount of attention [5,17]. We investigate this model because of its intrinsic interest and its deterministic construction, which allows one to study analytically its Laplacian spectra and their corresponding eigenvectors.

The deterministically growing trees, denoted by U_t ($t \geq 0$) after t iterations, are constructed as follows. For $t=0$, U_0 is an edge connecting two nodes. For $t \geq 1$, U_t is obtained from U_{t-1} by attaching m (m is a positive integer) new nodes to each node in U_{t-1}. Figure 1 illustrates the construction process of a particular network for the case of $m=2$ for the first four generations.

According to the network construction, one can see that at each step t_i ($t_i \geq 1$) the number of newly introduced nodes is $L(t_i)=2m(m+1)^{t_i-1}$. From this result, we can easily compute the network order (i.e., the total number of nodes) N_t at step t,

$$N_t = \sum_{t_i=0}^{t} L(t_i) = 2(m+1)^t. \tag{1}$$

The considered networks have a degree distribution of exponential form. Their cumulative degree distribution

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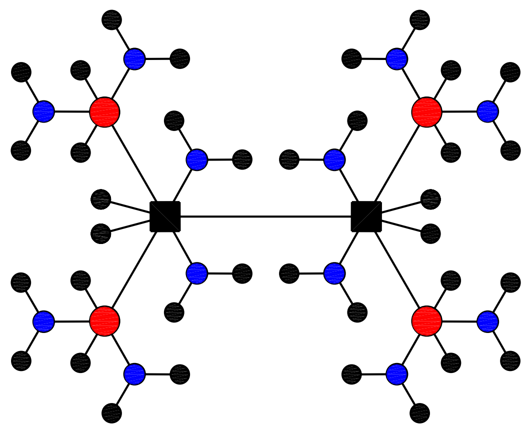

FIG. 1. (Color online) Illustration of a deterministic uniform recursive tree for the special case of $m=2$, showing the first several steps of growth process.

$P_{\mathrm{cum}}(k)$, defined to be the probability that the degree is greater than or equal to k, decays exponentially with k as $P_{\mathrm{cum}}(k)=(m+1)^{-(k-1)/m}$ [16]. Their average path length, defined as the mean of shortest distance between all pairs of nodes, increases logarithmically with network order [17]. Thus, the networks exhibit small-world behavior [18].

Notice that the particular case of $m=1$ is in fact a deterministic version of the uniform recursive tree (URT) [19], which is a principal model [20,21] of random graphs [22]. As one of the most widely studied model, the URT is constructed as follows [19]: start with a single node, at each time step, we attach a new node to an existing node selected at random. It has found many important applications in various areas. For example, it has been suggested as models for the spread of epidemics [23], the family trees of preserved copies of ancient or medieval texts [24], chain letter, and pyramid schemes [25], to name but a few. The $m=1$ case of the networks studied here has similar structural properties as the URT; thus, we call the considered networks expanded deterministic uniform recursive trees (EDURTs), which could shed light in better understanding the nature of the URT. After introducing the EDURTs, in what follows we will study the eigenvalues and their corresponding eigenvectors of the Laplacian matrices of the EDURTs.

III. LAPLACIAN SPECTRA AND THEIR CORRESPONDING EIGENVECTORS

Generally, for an arbitrary graph, it is difficult to determine all eigenvalues and eigenvectors of its Laplacian matrix, but below we will show that for U_t one can settle this problem.

A. Eigenvalues

As known in Eq. (1), there are $2(m+1)^t$ vertices in U_t. We denote by V_t the vertex set of U_t, i.e., $V_t=\{v_1,v_2,\ldots,v_{2(m+1)^t}\}$. Let $\mathbf{A}_t=[a_{ij}]$ be the adjacency matrix of network U_t, where $a_{ij}=a_{ji}=1$ if nodes i and j are connected, $a_{ij}=a_{ji}=0$ otherwise, then the degree of vertex v_i is defined as $d_{v_i}=\Sigma_{j\in V_t}a_{ij}$. Let $\mathbf{D}_t=\mathrm{diag}(d_{v_1},d_{v_2},\ldots,d_{v_{2(m+1)^t}})$ represent the diagonal degree matrix of U_t, then the Laplacian matrix of U_t is defined by $\mathbf{L}_t=\mathbf{D}_t-\mathbf{A}_t$.

We first study the Laplacian spectra of U_t, while we leave the eigenvectors to the next subsection. By construction, it is easy to find that the adjacency matrix $\mathbf{A}_t$ and diagonal degree matrix $\mathbf{D}_t$ satisfy the following relations:

$$\mathbf{A}_t=\begin{pmatrix} \mathbf{A}_{t-1} & \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}_{(m+1)\times(m+1)} \tag{2}$$

and

$$\mathbf{D}_t=\begin{pmatrix} \mathbf{D}_{t-1}+m\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \end{pmatrix}_{(m+1)\times(m+1)}, \tag{3}$$

where each block is a $2(m+1)^{t-1}\times 2(m+1)^{t-1}$ matrix and $\mathbf{I}$ is the identity matrix. Thus, according to the above expressions (for $\mathbf{A}_t$ and $\mathbf{D}_t$) and the definition of Laplacian matrix, we have the following recursive relation between $\mathbf{L}_t$ and $\mathbf{L}_{t-1}$:

$$\mathbf{L}_t=\mathbf{D}_t-\mathbf{A}_t=\begin{pmatrix} \mathbf{L}_{t-1}+m\mathbf{I} & -\mathbf{I} & -\mathbf{I} & \cdots & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \end{pmatrix}. \tag{4}$$

Then, the characteristic polynomial of $\mathbf{L}_t$ is

$$P_t(x)=\det(x\mathbf{I}-\mathbf{L}_t)=\det\begin{pmatrix}(x-m)\mathbf{I}-\mathbf{L}_{t-1} & \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I}\\ \mathbf{I} & (x-1)\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{I} & \mathbf{0} & (x-1)\mathbf{I} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots\\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & (x-1)\mathbf{I}\end{pmatrix}$$
$$=\{\det[(x-1)\mathbf{I}]\}^m\det\begin{pmatrix}(x-m)\mathbf{I}-\mathbf{L}_{t-1} & \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I}\\ \frac{1}{x-1}\mathbf{I} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0}\\ \frac{1}{x-1}\mathbf{I} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots\\ \frac{1}{x-1}\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}\end{pmatrix}$$
$$=\{\det[(x-1)\mathbf{I}]\}^m\det\begin{pmatrix}\left(x-m-\frac{m}{x-1}\right)\mathbf{I}-\mathbf{L}_{t-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \frac{1}{x-1}\mathbf{I} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0}\\ \frac{1}{x-1}\mathbf{I} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots\\ \frac{1}{x-1}\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}\end{pmatrix}, \quad (5)$$

where the elementary operations of matrix have been used. According to the results in [26], we have

$$P_t(x)=\{\det[(x-1)\mathbf{I}]\}^m\det\left[\left(x-m-\frac{m}{x-1}\right)\mathbf{I}-\mathbf{L}_{t-1}\right]. \quad (6)$$

Thus, $P_t(x)$ can be recast recursively as follows:

$$P_t(x)=(x-1)^{2m(m+1)^{t-1}}P_{t-1}(\varphi(x)), \quad (7)$$

where $\varphi(x)=x-m-\frac{m}{x-1}$. This recursive relation given by Eq. (7) is very important, from which we will determine the complete Laplacian eigenvalues of U_t and their corresponding eigenvectors. Notice that $P_{t-1}(x)$ is a monic polynomial of degree $2(m+1)^{t-1}$, then the exponent of $\frac{m}{x-1}$ in $P_{t-1}(\varphi(x))$ is $2(m+1)^{t-1}$, and hence the exponent of factor $x-1$ in $P_t(x)$ is

$$2m(m+1)^{t-1}-2(m+1)^{t-1}=2(m-1)(m+1)^{t-1}. \quad (8)$$

Consequently, 1 is an eigenvalue of $\mathbf{L}_t$, and its multiplicity is $2(m-1)(m+1)^{t-1}$.

Note that U_t has $2(m+1)^t$ Laplacian eigenvalues. We represent these $2(m+1)^t$ Laplacian eigenvalues as $\lambda_1^t,\lambda_2^t,\ldots,\lambda_{2(m+1)^t}^t$, respectively. For convenience, we presume $\lambda_1^t\le\lambda_2^t\le\cdots\le\lambda_{2(m+1)^t}^t$, and denote by E_t the set of these Laplacian eigenvalues, i.e., $E_t=\{\lambda_1^t,\lambda_2^t,\ldots,\lambda_{2(m+1)^t}^t\}$. All the Laplacian eigenvalues in set E_t can be divided into two parts. According to the above analysis, $\lambda=1$ is a Laplacian eigenvalue with multiplicity $2(m-1)(m+1)^{t-1}$, which gives a part of the eigenvalues of L_t. We denote by E_t' the set of Laplacian eigenvalues 1 of U_t, i.e.,

$$E_t'=\{\underbrace{1,1,1,\ldots,1,1}_{2(m-1)(m+1)^{t-1}}\}. \quad (9)$$

It should be noted that here we neglect the distinctness of elements in the set. The remaining $4(m+1)^{t-1}$ Laplacian eigenvalues of U_t are determined by the equation $P_{t-1}(\varphi(x))=0$. Let the $4(m+1)^{t-1}$ eigenvalues be $\tilde{\lambda}_1^t,\tilde{\lambda}_2^t,\ldots,\tilde{\lambda}_{4(m+1)^{t-1}}^t$, respectively. For convenience, we presume $\tilde{\lambda}_1^t\le\tilde{\lambda}_2^t\le\cdots\le\tilde{\lambda}_{4(m+1)^{t-1}}^t$ and denote by E_t^* the set of these eigenvalues, i.e., $E_t^*=\{\tilde{\lambda}_1^t,\tilde{\lambda}_2^t,\ldots,\tilde{\lambda}_{4(m+1)^{t-1}}^t\}$. Therefore, the set of all Laplacian eigenvalues for U_t can be expressed as $E_t=E_t'\cup E_t^*$.

According to Eq. (7), for an arbitrary element in E_{t-1}, say $\lambda_i^{t-1}\in E_{t-1}$, both solutions of $x-m-\frac{m}{x-1}=\lambda_i^{t-1}$ are in E_t^*. In fact, equation $x-m-\frac{m}{x-1}=\lambda_i^{t-1}$ is equivalent to

$$x^2-(\lambda_i^{t-1}+m+1)x+\lambda_i^{t-1}=0. \quad (10)$$

We use notations $\tilde{\lambda}_i^t$ and $\tilde{\lambda}_{i+2(m+1)^{t-1}}^t$ to represent the two solutions of Eq. (10), since they provide a natural increasing order of the eigenvalues of U_t, which can be seen from the argument below. Solving this quadratic equation, its roots are obtained to be $\tilde{\lambda}_i^t = r_1(\lambda_i^{t-1})$ and $\tilde{\lambda}_{i+2(m+1)^{t-1}}^t = r_2(\lambda_i^{t-1})$, where the functions $r_1(\lambda)$ and $r_2(\lambda)$ satisfy

$$r_1(\lambda) = \frac{1}{2}(\lambda + m + 1 - \sqrt{(\lambda + m + 1)^2 - 4\lambda}), \quad (11)$$

$$r_2(\lambda) = \frac{1}{2}(\lambda + m + 1 + \sqrt{(\lambda + m + 1)^2 - 4\lambda}). \quad (12)$$

Substituting each Laplacian eigenvalue of U_{t-1} into Eqs. (11) and (12), we can obtain the subset E_t^* of Laplacian eigenvalues of U_t. Since $E_0 = \{0,2\}$, by recursively applying the functions provided by Eqs. (11) and (12), the Laplacian spectra of U_t can be determined completely.

It is obvious that both $r_1(\lambda)$ and $r_2(\lambda)$ are monotonously increasing functions and that they lie in intervals $[0,1)$ and $(1,+\infty)$, respectively. On the other hand, since $r_1(\lambda)-1 = \frac{1}{2}(\lambda+m-1-\sqrt{(\lambda+m-1)^2+4m}) < 0$, we have $r_1(\lambda) < 1$. Similarly, we can show that $r_2(\lambda) > 1$. Thus for arbitrary fixed λ', $r_1(\lambda) < 1 < r_2(\lambda')$ holds for all λ. Then we have the following conclusion. If the set of Laplacian eigenvalues for U_{t-1} is $E_{t-1} = \{\lambda_1^{t-1}, \lambda_2^{t-1}, \ldots, \lambda_{2(m+1)^{t-1}}^{t-1}\}$, then solving Eqs. (11) and (12) one can obtain the subset E_t^* of Laplacian eigenvalues for U_t to be $E_t^* = \{\tilde{\lambda}_1^t, \tilde{\lambda}_2^t, \ldots, \tilde{\lambda}_{4(m+1)^{t-1}}^t\}$, where $\tilde{\lambda}_1^t \le \tilde{\lambda}_2^t \le \cdots \le \tilde{\lambda}_{2(m+1)^{t-1}}^t < 1 < \tilde{\lambda}_{2(m+1)^{t-1}+1}^t \le \tilde{\lambda}_{2(m+1)^{t-1}+2}^t \le \cdots \le \tilde{\lambda}_{4(m+1)^{t-1}}^t$. Recall that E_t' consists of $2(m-1)(m+1)^{t-1}$ elements, all of which are 1, so we can easily get the set of eigenvalue spectra for U_t to be $E_t = E_t^* \cup E_t'$.

From above arguments, it is easy to see that for the special case of $m=1$, all the 2^{t+1} Laplacian eigenvalues of U_t are fundamentally distinct, which is an interesting property and has never (to the best of our knowledge) been previously reported in other network models, thus may have some far-reaching consequences. For other cases $m > 1$, some eigenvalues (e.g., 1) are multiple, which is obviously different from that of $m=1$ case.

It has been established that Laplacian eigenvalues have connections with many contexts in the theory of networks. For example, they are closely related to the number of spanning trees on complex networks [27]. It has been shown that the number of spanning tress on a connected network G with order N, $N_{\mathrm{st}}(G)$, concerns with all its nonzero Laplacian eigenvalues λ_i (assuming $\lambda_1 = 0$ and $\lambda_i \neq 0$ for $i = 2, \ldots, N$), obeying the following expression [28]:

$$N_{\mathrm{st}}(G) = \frac{1}{N}\prod_{i=2}^{N} \lambda_i. \quad (13)$$

Since U_t are trees for all parameter m, according to Eq. (13), the product of all nonzero Laplacian eigenvalues for U_t, denoted by Λ_t, should be equal to N_t, which can be confirmed from the following argument. For $t=0$, by construction it is obvious that $\Lambda_0 = N_0 = 2$; for $t \ge 1$, according to Eq. (10), we can easily obtain the following recursive relation $\Lambda_t = (m+1)\Lambda_{t-1}$, which combining with the initial value $\Lambda_0 = 2$ leads to $\Lambda_t = 2(m+1)^t = N_t$. This proves that our computation on the Laplacian eigenvalues for U_t is right.

B. Eigenvectors

Similar to the eigenvalues, the eigenvectors of $\mathbf{L}_t$ follow directly from those of $\mathbf{L}_{t-1}$. Assume that λ is an arbitrary Laplacian eigenvalue of U_t, whose corresponding eigenvector is $\boldsymbol{v} \in \mathbf{R}^{2(m+1)^t}$, where $\mathbf{R}^{2(m+1)^t}$ represents the $2(m+1)^t$-dimensional vector space. Then we can solve equation $(\lambda\mathbf{I} - \mathbf{L}_t)\boldsymbol{v} = 0$ to find the eigenvector $\boldsymbol{v}$. We distinguish two cases: $\lambda \in E_t^*$ and $\lambda \in E_t'$, which will be separately addressed in detail as follows.

For the first case $\lambda \in E_t^*$, we can rewrite the equation $(\lambda\mathbf{I} - \mathbf{L}_t)\boldsymbol{v} = 0$ as

$$\begin{pmatrix} (\lambda - m)\mathbf{I} - \mathbf{L}_{t-1} & \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \\ \mathbf{I} & (\lambda - 1)\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & (\lambda - 1)\mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & (\lambda - 1)\mathbf{I} \end{pmatrix} \times \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \vdots \\ \boldsymbol{v}_{m+1} \end{pmatrix} = 0, \quad (14)$$

where vector $\boldsymbol{v}_i$ $(1 \le i \le m+1)$ are components of $\boldsymbol{v}$. Equation (14) results in the following equations:

$$[(\lambda - m)\mathbf{I}_{t-1} - \mathbf{L}_{t-1}]\boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_{m+1} = \mathbf{0}, \quad (15)$$

$$\boldsymbol{v}_1 + (\lambda - 1)\boldsymbol{v}_i = \mathbf{0} \quad (2 \le i \le m+1). \quad (16)$$

Resolving Eq. (16), we find that

$$\boldsymbol{v}_i = -\frac{1}{\lambda - 1}\boldsymbol{v}_1 \quad (2 \le i \le m+1). \quad (17)$$

Substituting Eq. (17) into Eq. (15) we have

$$\left[\left(\lambda - m - \frac{m}{\lambda - 1}\right)\mathbf{I} - \mathbf{L}_{t-1}\right]\boldsymbol{v}_1 = 0, \quad (18)$$

which indicates that $\boldsymbol{v}_1$ is the solution of Eq. (15) while $\boldsymbol{v}_i$ $(2 \le i \le m+1)$ are uniquely decided by $\boldsymbol{v}_1$ via Eq. (17).

In Eq. (7), it is clear that if λ is an eigenvalue of Laplacian matrix $\mathbf{L}_t$, then $f(\lambda) = \lambda - m - \frac{m}{\lambda-1}$ must be one eigenvalue of $\mathbf{L}_{t-1}$. [Recall that if $\lambda = \tilde{\lambda}_i^t \in E_t^*$, then $\varphi(\tilde{\lambda}_i^t) = \lambda_i^{t-1}$ for $i \le 2(m+1)^{t-1}$ or $\varphi(\tilde{\lambda}_i^t) = \lambda_{i-2(m+1)^{t-1}}$ for $i > 2(m+1)^{t-1}$.] Thus, Eq. (18) together with Eq. (7) shows that $\boldsymbol{v}_1$ is an eigenvector of matrix $\mathbf{L}_{t-1}$ corresponding to the eigenvalue $\lambda - m - \frac{m}{\lambda-1}$ determined by λ, while

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \vdots \\ \boldsymbol{v}_{m+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1 \\ -\frac{1}{\lambda-1}\boldsymbol{v}_1 \\ -\frac{1}{\lambda-1}\boldsymbol{v}_1 \\ \vdots \\ -\frac{1}{\lambda-1}\boldsymbol{v}_1 \end{pmatrix} \quad (19)$$

is an eigenvector of $\mathbf{L}_t$ corresponding to the eigenvalue λ.

Since for the initial graph U_0, its Laplacian matrix $\mathbf{L}_0$ has two eigenvalues 0 and 2 with respective eigenvectors $(1,1)^\top$ and $(1,-1)^\top$; by recursively applying the above process, we can obtain all the eigenvectors corresponding to $\lambda \in E_t^*$.

For the second case of $\lambda \in E_t'$, where all $\lambda=1$, the equation $(\lambda\mathbf{I}-\mathbf{L}_t)\boldsymbol{v}=0$ can be recast as

$$\begin{pmatrix} (1-m)\mathbf{I}-\mathbf{L}_{t-1} & \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \vdots \\ \boldsymbol{v}_{m+1} \end{pmatrix} = 0, \quad (20)$$

where vector $\boldsymbol{v}_i$ $(1 \leq i \leq m+1)$ are components of $\boldsymbol{v}$. Equation (20) leads to the following equations:

$$\boldsymbol{v}_1 = \mathbf{0}, \quad (21)$$

$$\boldsymbol{v}_2 + \boldsymbol{v}_3 + \cdots + \boldsymbol{v}_{m+1} = \mathbf{0}. \quad (22)$$

In Eq. (21), $\boldsymbol{v}_1$ is a zero vector, and we denote by $\boldsymbol{v}_{i,j}$ the jth component of the column vector $\boldsymbol{v}_i$. On the other hand, Eq. (22) gives us the following equations:

$$\begin{cases} \boldsymbol{v}_{2,1} + \boldsymbol{v}_{3,1} + \cdots + \boldsymbol{v}_{m+1,1} = \mathbf{0} \\ \boldsymbol{v}_{2,2} + \boldsymbol{v}_{3,2} + \cdots + \boldsymbol{v}_{m+1,2} = \mathbf{0} \\ \quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \boldsymbol{v}_{2,2(m+1)^{t-1}} + \boldsymbol{v}_{3,2(m+1)^{t-1}} + \cdots + \boldsymbol{v}_{m+1,2(m+1)^{t-1}} = \mathbf{0} \end{cases}$$

The set of all solutions to any of the above equations consists of vectors that can be written as

$$\begin{pmatrix} \boldsymbol{v}_{2,j} \\ \boldsymbol{v}_{3,j} \\ \boldsymbol{v}_{4,j} \\ \vdots \\ \boldsymbol{v}_{m+1,j} \end{pmatrix} = k_{1,j} \begin{pmatrix} -\mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} + k_{2,j} \begin{pmatrix} -\mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \vdots \\ \mathbf{0} \end{pmatrix} + \cdots + k_{m-\mathbf{1},j} \begin{pmatrix} -\mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{1} \end{pmatrix}, \quad (23)$$

where $k_{1,j}, k_{2,j}, \ldots, k_{m-1,j}$ are arbitrary real numbers. In Eq. (23), the solutions for all the vectors $\boldsymbol{v}_i$ $(2 \leq i \leq m+1)$ can be rewritten as

$$\begin{pmatrix} \boldsymbol{v}_2^\top \\ \boldsymbol{v}_3^\top \\ \boldsymbol{v}_4^\top \\ \vdots \\ \boldsymbol{v}_{m+1}^\top \end{pmatrix} = \begin{pmatrix} -\mathbf{1} & -\mathbf{1} & \cdots & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{pmatrix} \times \begin{pmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,2(m+1)^{t-1}} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,2(m+1)^{t-1}} \\ k_{3,1} & k_{3,2} & \cdots & k_{3,2(m+1)^{t-1}} \\ \vdots & \vdots & & \vdots \\ k_{m-1,1} & k_{m-1,2} & \cdots & k_{m-1,2(m+1)^{t-1}} \end{pmatrix}, \quad (24)$$

where $k_{i,j}$ $(1 \leq i \leq m-1; 1 \leq j \leq 2(m+1)^{t-1})$ are arbitrary real numbers. According to Eq. (24), we can obtain the eigenvector $\boldsymbol{v}$ corresponding to the eigenvalue **1**. Moreover, it is easy to see that the dimension of the eigenspace of matrix $\mathbf{L}_t$ associated with eigenvalue **1** is $2(m-1)(m+1)^{t-1}$.

In this way, all eigenvalues and their corresponding eigenvectors of U_t have been completely determined in a recursive way.

IV. CONCLUSIONS

In this paper, we have investigated the Laplacian eigenvalues and their corresponding eigenvectors of a family of deterministically growing treelike networks that exhibit small-world behavior. Making use of the methods of linear algebra and graph theory, we have fully characterized the Laplacian eigenvalues and eigenvectors of the networks, all of which are recursively determined from those for the initial network. Interestingly, we showed that for a particular case $(m=1)$ of the networks under consideration, all its Laplacian eigenvalues are disparate. We expect our results to be interesting in some fields of networks, such as random and quantum walks on networks, the computation of the resistance between two arbitrary nodes in a resistor network, the dynamics of coupled oscillators on networks, and so on. We also expect that the computing methods of eigenvalues and eigenvectors used here might be extended to other types deterministic networks, e.g., deterministic small-world networks [29,30] and deterministic scale-free networks [31–36].

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